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Advances in Applied Mathematics 31 (2003) 199–214
ADVANCES IN Applied Mathematics
www.elsevier.com/locate/yaama

Particle seas and basic hypergeometric series

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Received 14 March 2002; accepted 28 September 2002

Abstract

Particle seas were introduced by Claude Itzykson to give a direct combinatorial proof of the Jacobi triple product [Viennot, Empilements, 1999]. We show here how generalized particle seas can be employed to give bijective proofs of several identities in the theory of basic hypergeometric series. We give details for the ${}_1\psi_1$ summation and two limiting cases of the ${}_6\psi_5$ summation.

1. Introduction

Recently [4] we demonstrated how Ramanujan's ${}_1\psi_1$ summation is equivalent to a combinatorial statement about certain types of partitions and proved this statement bijectively. We first wrote the ${}_1\psi_1$ in the form

$$\frac{(-aq;q)_\infty(-bq;q)_\infty}{(q;q)_\infty(abq;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-a^{-1};q)_n(zqa)^n}{(-bq;q)_n}=\frac{(-zq;q)_\infty(-z^{-1};q)_\infty}{(bz^{-1};q)_\infty(azq;q)_\infty}, \tag{1.1}$$

where $|b|<|z|<|1/(aq)|$ and $|q|<1$, and where we employ the standard notation

$$(a;q)_\infty=\prod_{k=0}^{\infty}\left(1-aq^k\right) \quad \text{and} \quad (a;q)_n=\frac{(a;q)_\infty}{(aq^n;q)_\infty}. \tag{1.2}$$

Then we showed that the coefficients of z^0 are equal by exhibiting a bijection between certain Frobenius partitions and 4-tuples of partitions. Finally we used elementary

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0196-8858/03/$ – see front matter
doi:10.1016/S0196-8858(03)00011-3

analytical arguments to complete the proof, which is a generalization of a proof for the Jacobi Triple product [2,9].

Here we shall introduce an alternative construction to the Frobenius partitions in [4], called particle seas. The first advantage of this new viewpoint will be that for any $m \in \mathbb{Z}$, we can identify a simple bijection which proves the equality of coefficients of z^m on both sides of (1.1). The second application of the particle seas will be to the ${}_6\phi_5$ summation [7], which we write

$$\sum_{n=0}^{\infty} \frac{(1+dq^{2n})(-dq;q)_{n-1}(-1/b;q)_n(-1/c;q)_n(-d/a;q)_n(abcq)^n}{(q;q)_n(bdq;q)_n(cdq;q)_n(aq;q)_n}$$
$$= \frac{(-acq;q)_\infty(-abq;q)_\infty}{(aq;q)_\infty(cabq;q)_\infty} \frac{(-dq;q)_\infty(-dcbq;q)_\infty}{(dcq;q)_\infty(dbq;q)_\infty}. \tag{1.3}$$

When $d = 0$ we obtain a q-analogue of a theorem of Gauss,

$$\sum_{n=0}^{\infty} \frac{(-1/b;q)_n(-1/c;q)_n(cabq)^n}{(q;q)_n(aq;q)_n} = \frac{(-acq;q)_\infty(-abq;q)_\infty}{(aq;q)_\infty(cabq;q)_\infty}, \tag{1.4}$$

and setting $a = 0$ we have

$$\sum_{n=0}^{\infty} \frac{(1+dq^{2n})(-dq;q)_{n-1}(-1/b;q)_n(-1/c;q)_n(bc)^n q^{n(n+1)/2}}{(q;q)_n(bdq;q)_n(cdq;q)_n}$$
$$= \frac{(-dq;q)_\infty(-dcbq;q)_\infty}{(dcq;q)_\infty(dbq;q)_\infty}. \tag{1.5}$$

We shall exhibit straightforward combinatorial proofs of (1.4) and (1.5) using particle seas.

The particle seas were introduced by Claude Itzykson to give a straightforward proof of the triple product identity. We learned that proof in an amazing graduate class of Xavier Viennot [13]. To our knowledge this proof was never published, so we shall include it here. It is equivalent to the Hathaway–Sylvester proof rewritten by Wright [3,11]. Other combinatorial proofs of the Jacobi Triple product can be found in [6,8]. We present overpartitions [5] which will help to introduce the particle seas. After discussing some generating functions and combinatorics, we shall use the particle seas to prove the basic hypergeometric series identities.

2. Overpartitions

A partition into k positive (respectively nonnegative) parts is a sequence $\lambda = (\lambda_1, \dots, \lambda_k)$ such that $\lambda_1 \geqslant \cdots \geqslant \lambda_k > 0$ (respectively $\lambda_1 \geqslant \cdots \geqslant \lambda_k \geqslant 0$). Let P (respectively $P_{\geqslant}$) be the set of partitions into positive (respectively nonnegative) parts. Let P_n (respectively $P_{n,\geqslant}$) be the set of partitions into positive (respectively nonnegative) parts less or equal to n.

A partition into k distinct positive parts is a sequence $\lambda = (\lambda_1, \dots, \lambda_k)$ such that $\lambda_1 > \cdots > \lambda_k > 0$. Let D (respectively $D_{\geqslant}$) be the set of partitions into positive (respectively nonnegative) parts. Let D_n (respectively $D_{n,\geqslant}$) be the set of partitions into distinct positive (respectively nonnegative) parts less or equal to n. We denote by $l(\lambda)$ the number of parts of λ and by $|\lambda|$ the weight of the partition. In other words, $l(\lambda) = k$ and $|\lambda| = \lambda_1 + \cdots + \lambda_k$. We denote by $m_\lambda(i)$ the multiplicity of the part i in λ.

The following classical results about generating functions of partitions can be found in [1, Chapter 1]:

$$\sum_{\lambda \in P} q^{|\lambda|} z^{l(\lambda)} = \frac{1}{(zq;q)_\infty}, \qquad \sum_{\lambda \in P_{\geqslant}} q^{|\lambda|} z^{l(\lambda)} = \frac{1}{(z;q)_\infty},$$

$$\sum_{\lambda \in P_n} q^{|\lambda|} z^{l(\lambda)} = \frac{1}{(zq;q)_n}, \qquad \sum_{\lambda \in D} q^{|\lambda|} z^{l(\lambda)} = (-zq;q)_\infty,$$

$$\sum_{\lambda \in D_{\geqslant}} q^{|\lambda|} z^{l(\lambda)} = (-z;q)_\infty, \qquad \sum_{\lambda \in D_n} q^{|\lambda|} z^{l(\lambda)} = (-zq;q)_n.$$

We are now prepared to introduce overpartitions.

Definition 1. An *overpartition* is a partition such that the first occurrence of a part can be overlined.

Remark. These objects were introduced by MacMahon who chose to possibly distinguish the last occurrence of a part [9,10]. Some of their properties are developed in [5].

Let O (respectively $O_{\geqslant}$) denote the set of overpartitions into positive (respectively nonnegative) parts and let O_n (respectively $O_{n,\geqslant}$) be the set of overpartitions into positive (respectively nonnegative) parts with largest part at most n. If $\lambda \in O$ let $o(\lambda)$ be its number of overlined parts. As the overlined parts in the overpartition form a partition into distinct parts and the others form an ordinary partition, the following generating functions are straightforward:

Lemma 2. *The generating functions for overpartitions are*

$$\sum_{\lambda \in O} q^{|\lambda|} z^{l(\lambda)} x^{o(\lambda)} = \frac{(-xzq;q)_\infty}{(zq;q)_\infty}, \qquad \sum_{\lambda \in O_{\geqslant}} q^{|\lambda|} z^{l(\lambda)} x^{o(\lambda)} = \frac{(-xz;q)_\infty}{(z;q)_\infty}, \tag{2.1}$$

$$\sum_{\lambda \in O_n} q^{|\lambda|} z^{l(\lambda)} x^{o(\lambda)} = \frac{(-xzq;q)_n}{(zq;q)_n}, \qquad \sum_{\lambda \in O_{n,\geqslant}} q^{|\lambda|} z^{l(\lambda)} x^{o(\lambda)} = \frac{(-xz;q)_{n+1}}{(z;q)_{n+1}}. \tag{2.2}$$

Let $\mathcal{O}_k$ (respectively $\mathcal{O}_{k,\geqslant}$) be the set of overpartitions into k positive (respectively nonnegative) parts, and write $c(\lambda)$ for the number of overlined parts less than the largest part which could occur in the partition but do not.

Lemma 3.

$$\sum_{\lambda\in\mathcal{O}_k} q^{|\lambda|}x^{o(\lambda)}d^{c(\lambda)} = \frac{q^k(-x;q)_k}{(dq;q)_k}, \qquad \sum_{\lambda\in\mathcal{O}_{k,\geqslant}} q^{|\lambda|}x^{o(\lambda)}d^{c(\lambda)} = \frac{(-x;q)_k}{(dq;q)_k}. \tag{2.3}$$

Proof. The second identity with $d=1$ was considered in [12]. Their argument applies for arbitrary d as well, so we repeat it here. We need to define a bijection ψ_k between $D_{k,\geqslant}\times P_k$ and $\mathcal{O}_{k,\geqslant}$. This bijection must be such that if $\alpha\in D_{k,\geqslant}$, $\mu\in P_k$, and $\lambda=\psi_k(\alpha,\mu)$ then

$$|\lambda| = |\alpha|+|\mu|, \qquad o(\lambda)=l(\alpha), \qquad c(\lambda)=l(\mu).$$

Starting with $\alpha\in D_{k,\geqslant}$ and $\mu\in P_k$ let us define $\lambda=\psi_k(\alpha,\mu)$ by

$$\lambda_i = \bigl|\{j\colon \mu_j\geqslant i\}\bigr| + \bigl|\{j\colon \alpha_j\geqslant i\}\bigr|, \qquad \lambda_i \text{ overlined iff } (i-1)\in\alpha, \quad 1\leqslant i\leqslant k. \qquad \square$$

We note, as was done in [12], that Lemmas 2 and 3 with $d=1$ give two different ways to enumerate the overpartitions, and hence we have a bijective proof of

$$\sum_{k=0}^{\infty}\frac{(-x;q)_k z^k q^k}{(q;q)_k} = \frac{(-xzq;q)_\infty}{(zq;q)_\infty},$$

which is the classical q-binomial theorem.

3. Particle seas

In this section we will use the discrete plane, in particular its nonnegative half (i.e., the set of points (x,y) with $x\in\mathbb{Z}$ and $y\in\mathbb{N}$). We partially fill that half discrete plane with (colored) squares and balls to create the particle seas. The zero-line is the sets of points $(x,0)$ with $x\in\mathbb{Z}$. The positive (respectively nonpositive) quarter is the set of points (x,y) with $x\in\mathbb{N}^+$ (respectively $x\in\mathbb{Z}\backslash\mathbb{N}^+$) and $y\in\mathbb{N}$.

Definition 4. A particle sea s is the upper half discrete plane partly filled with squares and balls such that:

- the zero-line is partially filled with squares or balls,
- the rest of the positive quarter is partially filled with squares,
- the rest of the nonpositive quarter is partially filled with balls,
- if the point (x,y) with $y>0$ is filled then the point $(x,y-1)$ is filled too,
- if there is a ball (respectively square) of positive (respectively nonpositive) abscissa x then there is a square (respectively ball) of positive (respectively nonpositive) abscissa x' with $|x'|\geqslant|x|$.

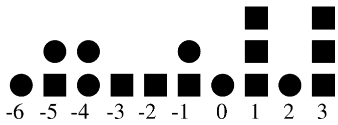

Fig. 1. Example of a particle sea.

The *weight* of a particle sea s, denoted by $|s|$, is the difference between the sum of the abscissas of the squares in the positive quarter and the sum of the abscissas of the balls in the nonpositive quarter. Let $d(s)$ be the number of squares in the positive quarter and $g(s)$ the number of balls in the nonpositive quarter. Let $d_0(s)$ (respectively $g_0(s)$) be the number of squares (respectively balls) with ordinate 0 and positive (respectively nonpositive) abscissa. Let $w_d(s)$ (respectively $-w_g(s)$) be the largest (respectively smallest) abscissa x where there exists square (respectively ball) with abscissa x. Let $t_g(s)$ be the number of squares with abscissa $x \leqslant 0$.

In the example on Fig. 1 we have a particle sea s with $|s| = 32$, $g_0(s) = 3$, $w_g(s) = 6$, $g(s) = 6$, $d_0(s) = 2$, $w_d(s) = 3$, $d(s) = 6$, and $t_g(s) = 4$.

From this definition it is easy to construct a bijection between particle seas and pairs of overpartitions, $\phi : S \to O \times O_{\geqslant}$.

Algorithm.

$\phi : s \mapsto (\lambda, \mu)$
For i from 1 to $w_d(s)$ do
  $m_\lambda(i) \leftarrow$ number of squares of abscissas i
  If s has a square on $(i, 0)$ then
    Overline the first part i in λ
For i from 0 to $w_g(s)$
  $m_\mu(i) \leftarrow$ number of balls of abscissas $-i$
  If s has a ball on $(-i, 0)$ then
    Overline the first part i in μ

Let $o(\lambda)$ be the number of overlined parts in λ. Then it is clear from this bijection that $|s| = |\lambda| + |\mu|$ and

$$d(s) = l(\lambda), \qquad d_0(s) = o(\lambda), \qquad w_d(s) = \lambda_1, \qquad g(s) = l(\mu),$$
$$g_0(s) = o(\mu), \qquad w_g(s) = \mu_1, \qquad t_g(s) = c(\mu). \tag{3.1}$$

For the particle sea in Fig. 1 we get $\lambda = (\bar{3}, 3, 3, \bar{1}, 1, 1)$ and $\mu = (\bar{6}, 5, \bar{4}, 4, 1, \bar{0})$.

Since pairs of overpartitions in $\mathcal{O}_k \times \mathcal{O}_{k,\geqslant}$ correspond to a type of Frobenius partition, the particle seas can be seen as Frobenius partitions except that the lengths of the two rows need not be the same [2].

Now we can give some results about generating function of particle seas. The proofs of these results are straightforward thanks to the bijection ϕ. Let S be the set of particle seas. Using Eq. (2.1):

$$\sum_{s\in S} q^{|s|}x^{d(s)}y^{g(s)}a^{d_0(s)}b^{g_0(s)} = \frac{(-axq;q)_\infty(-by;q)_\infty}{(xq;q)_\infty(y;q)_\infty}. \tag{3.2}$$

Let $S_{n,m}$ be the subset of S such that $s\in S_{n,m}$ if and only if $w_g(s)\leqslant n$ and $w_d(s)\leqslant m$. Then using (2.2):

$$\sum_{s\in S_{n,m}} q^{|s|}x^{d(s)}y^{g(s)}a^{d_0(s)}b^{g_0(s)} = \frac{(-axq;q)_m(-by;q)_n}{(xq;q)_m(y;q)_n}.$$

Let $\overline{S}_{n,m}$ be the subset of S such that $s\in \overline{S}_{n,m}$ if and only if $g(s)=n$ and $d(s)=m$.

Lemma 5.

$$\sum_{s\in \overline{S}_{n,m}} q^{|s|}a^{g(s)-g_0(s)}b^{d(s)-d_0(s)}c^{t_g(s)} = \frac{(-1/a;q)_n(-1/b;q)_m a^n b^m q^m}{(cq;q)_n(q;q)_m}. \tag{3.3}$$

Proof. Using Eqs. (3.1) and (2.3) we get the result. □

Let $\widetilde{S}_m$ be the subset of S such that $s\in\widetilde{S}_m$ if and only if $g(s)-d(s)=m$.

Lemma 6. *For $m\geqslant 0$,*

$$\sum_{s\in \widetilde{S}_m} q^{|s|}a^{g(s)-g_0(s)}b^{d(s)-d_0(s)}c^{d(s)+t_g(s)} = \sum_{n=0}^{\infty}\frac{(-1/a;q)_{n+m}(-1/b;q)_n(bcq)^n a^{n+m}}{(cq;q)_{m+n}(q;q)_n}, \tag{3.4}$$

$$\sum_{s\in \widetilde{S}_{-m}} q^{|s|}a^{g(s)-g_0(s)}b^{d(s)-d_0(s)}c^{d(s)+t_g(s)} = \sum_{n=0}^{\infty}\frac{(-1/a;q)_n(-1/b;q)_{n+m}(aq)^n (cb)^{n+m}}{(cq;q)_n(q;q)_{n+m}}. \tag{3.5}$$

Proof. Using Eqs. (3.1) and (2.3) we get the result. □

Let $S^*_{n,m}$ be the subset of $\overline{S}_{n,m}$ such that $s\in S^*_{n,m}$ if and only if $w_g(s)+1=g_0(s)$. Then it is straightforward to get, using (2.3):

$$\sum_{s\in S^*_{n,m}} q^{|s|}a^{g_0(s)}b^{d_0(s)} = \frac{q^m(-a;q)_n(-b;q)_m}{(q;q)_m}.$$

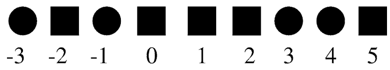

Fig. 2. Example of a flat sea particle.

Let F be the set of flat particle seas (that is only the line with ordinate zero can be filled). See Fig. 2. The bijection ϕ induces a bijection between flat particle seas and pairs of partitions into distinct parts (where one of them is in nonnegative parts). Therefore,

$$\sum_{s\in F} q^{|s|}x^{d(s)}y^{g(s)} = (-xq;q)_\infty(-y;q)_\infty. \tag{3.6}$$

Let H be the set of *heap particle seas* (that is a column can be made only of squares or only of balls). The bijection ϕ induces a bijection between heap particle seas and pairs of partitions (where one of them is in nonnegative parts). Therefore,

$$\sum_{s\in H} q^{|s|}x^{d(s)}y^{g(s)} = \frac{1}{(xq;q)_\infty(y;q)_\infty}. \tag{3.7}$$

Let $\mathcal{H}_{n,m}$ be the subset of H such that $s \in \mathcal{H}_{n,m}$ if and only if $g(s) = n$ and $d(s) = m$. Then

$$\sum_{s\in\mathcal{H}_{n,m}} q^{|s|}a^{w_g(s)}b^{w_d(s)} = \frac{bq^m}{(aq;q)_n(bq;q)_m}.$$

Now we define a new family of particle seas that will be useful to prove Eq. (1.5).

Definition 7. The *colored-square* seas are such that

- The squares can be black, white, gray or dark gray.
- The black squares do not appear on the zero-line and they appear in the positive quarter. The other squares and the balls only appear on the zero-line. Two white squares cannot appear consecutively.
- In the nonpositive quarter, going from left to right, the squares located between two balls are possibly gray then dark gray and finally white.
- If a point (x, y) with $y > 0$ is filled then $(x, y-1)$ is filled too.

For an example of a colored-square sea, see Fig. 3. As before the *weight* of a colored-square particle sea s, denoted by $|s|$, is the difference between the sum of the abscissas of the squares in the positive quarter and the sum of the abscissas of the balls in the

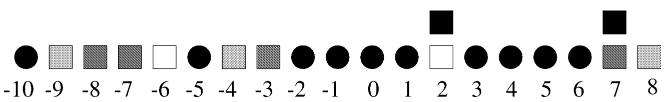

Fig. 3. Example of a colored-square sea.

nonpositive quarter; $d(s)$ is the number of squares in the positive quarter; $g(s)$ is the number of balls in the nonpositive quarter and $t_g(s)$ be the number of squares with abscissa $x \leqslant 0$.

Let $C_{m,n}$ be the set of *colored-square* seas such that $s \in C_{m,n}$ if and only if $g(s) = m$ and $d(s) = n$. Given a sea s in $C_{m,n}$ we define $b(s)$ the number of black squares, $w(s)$ the number of white squares, $gg(s)$ the number of gray squares and $dg(s)$ the number of dark gray squares.

Lemma 8.

$$\sum_{s \in C_{m,n}} q^{|s|} d^{t_g(s)} b^{b(s)+dg(s)} c^{b(s)+gg(s)}$$
$$= \frac{(-dq;q)_{m-1}(1+dq^{m+n})q^{m(m-1)/2}}{(bdq;q)_m(cdq;q)_m} \frac{(-1/b;q)_n(-1/c;q)_n(qbc)^n}{(q;q)_n}. \tag{3.8}$$

Proof. We first prove the second part of the right-hand side. We give a bijection between $D_{n-1,\geqslant} \times D_{n-1,\geqslant} \times P_n$ and the positive part of the sea (the points (x,y) with $x > 0$). Starting with $\alpha \in D_{n-1,\geqslant}$, $\beta \in D_{n-1,\geqslant}$ and $\gamma \in P_n$. We create λ:

$$\lambda_i = 1 + \bigl|\{j\colon \alpha_j \geqslant i\}\bigr| + \bigl|\{j\colon \beta_j \geqslant i\}\bigr| + \bigl|\{j\colon \gamma_j \geqslant i\}\bigr|,$$
$$\lambda_i \text{ is} \begin{cases} \text{overlined} & \text{iff } (i-1) \in \alpha, \\ \text{tilded} & \text{iff } (i-1) \in \beta, \end{cases} \quad 1 \leqslant i \leqslant n.$$

Then from λ we create the positive part of the sea. For i from 1 to n each part λ_i is transformed in a square of abscissa λ_i. This square is on the zero-line if the part is overlined or tilded. Then this square is white if the part is overlined and tilded, dark gray if the part is overlined, gray if it is tilded. Otherwise the square is not on the zero-line and it is black. Finally for i from 1 to n add balls on the points $(x, 0)$, $\lambda_i > x > \lambda_{i+1}$ (we consider that $\lambda_{n+1} = 0$).

Example. Let $n = 5$. Starting with $\alpha = (3,1)$, $\beta = (3,0)$, and $\gamma = (4,4,4)$ we get $\lambda = (\tilde{8}, \bar{7}, 7, \tilde{\bar{2}}, 2)$ which creates the positive part of the sea on Fig. 3.

Now we prove the first part of the right-hand side. We give a bijection between $D_m \times P_m \times P_m$ and the nonpositive part of the particle sea. Starting with $\alpha \in D_m$, $\beta \in P_m$ and $\gamma \in P_m$. We create a weighted sequence $\lambda = (\lambda_1, \ldots, \lambda_m)$ with weight $W(\lambda) = (W(\lambda_1), \ldots, (\lambda_m))$, $W(\lambda_i) \in \mathbb{N} \times \mathbb{N}$:

$$\lambda_i = -m + i - \bigl|\{j\colon \alpha_j \geqslant i\}\bigr| - \bigl|\{j\colon \beta_j \geqslant i\}\bigr| - \bigl|\{j\colon \gamma_j \geqslant i\}\bigr|,$$
$$W(\lambda_i) = \bigl(m_\beta(i), m_\gamma(i)\bigr), \quad 1 \leqslant i \leqslant m.$$

Note that the sequence λ is an increasing sequence of nonpositive numbers. Then from λ create the nonpositive part of the sea. For i from 1 to m each part λ_i is transformed into a ball of coordinates $(\lambda_i, 0)$. Add squares on the points $(x, 0)$, $\lambda_i < x < \lambda_{i+1}$ (we consider that $\lambda_{m+1} = 1$). Then if the weight of λ_i is (j, k) times then first j squares are gray then the following k squares are dark gray and if there is still one uncolored square it is colored white. Finally if $\alpha_1 = m$ then the right part of the sea is translated by 1 and a ball is added at the point $(1, 0)$. This guarantees that there will never be two consecutive white squares.

Example. Let $m = 5$. Starting with $\alpha = (1)$, $\beta = (2, 1)$, and $\gamma = (2, 1, 1)$ we get $\lambda = (-10, -5, -2, -1, 0)$ and $W(\lambda) = ((1, 2), (1, 1), (0, 0), (0, 0), (0, 0))$ which creates the nonpositive part of the sea on Fig. 3. □

4. Jacobi Triple product

Let us first write the Jacobi Triple product identity:

$$(-zq; q)_\infty \left(-z^{-1}; q\right)_\infty = \frac{1}{(q; q)_\infty} \sum_{n=-\infty}^{\infty} z^n q^{n(n+1)/2}.$$

We now describe the combinatorial proof of Claude Itzykson. Using the flat particle sea generating function (3.6) this identity says that there is a bijection $\Phi : F \to P \times \mathbb{Z}$. The bijection is such that if

$$(\alpha, n) = \Phi(s) \quad \text{then} \quad |s| = |\alpha| + \binom{d(s) - g(s) + 1}{2} \quad \text{and} \quad n = d(s) - g(s).$$

The bijection and its reverse are now defined:

Algorithm.

$\Phi : s \mapsto (\alpha, n)$
$n \leftarrow d(s) - g(s)$
Translate the integer line by $-n$
While there is a ball in the nonpositive half line
  Find the first square at its right
  $i \leftarrow$ difference of the abscissas
  Add a part i to α
  Delete the ball
  Change the square into a ball

$\Phi^{-1} : (\alpha, n) \mapsto s$
For i from 1 to $\alpha_1 - 1$ do
  Fill $(i, 0)$ with a ball
For i from 0 to $l(\alpha) - 1$ do
  Fill $(\alpha_{i+1} - i, 0)$ with a square
  Fill $(-i, 0)$ with a ball
Translate the integer line by n

A careful look at the algorithm shows that translating the line by $-n$ decreases the weight of the sea by $n(n + 1)/2$ and that α is a nonincreasing sequence.

Example. Starting with the sea s on Fig. 2 with $|s| = 12$, $g(s) = 2$, and $d(s) = 3$. We apply Φ and get $\alpha = (4, 2, 2, 2, 1)$ and $n = 1$. See Fig. 4.

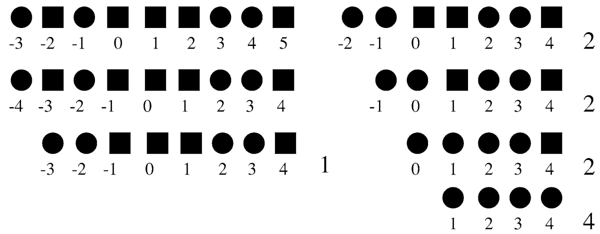

Fig. 4. Example of the bijection for Jacobi.

5. q-Gauss

The identity (1.4) is:

$$\sum_{n=0}^{\infty} \frac{(-1/b;q)_n(-1/c;q)_n(cabq)^n}{(q;q)_n(aq;q)_n} = \frac{(-acq;q)_\infty(-abq;q)_\infty}{(aq;q)_\infty(cabq;q)_\infty}.$$

Using Lemma 5 the identity says that there is a bijection $\Delta_n : \overline{S}_{n,n} \to P \times D \times D \times P$ such that if $s \mapsto (\alpha, \beta, \gamma, \delta)$ then

$$\begin{aligned} &|\alpha| + |\beta + |\gamma| + |\delta| = |s|, && l(\alpha) + l(\beta) + l(\gamma) + l(\delta) = n + t_g(s), \\ &l(\beta) + l(\delta) = n - g_0(s), && l(\gamma) + l(\delta) = n - d_0(s). \end{aligned} \tag{5.1}$$

This bijection, which was first applied in [4], is now explained using the language of particle seas. We begin by taking a particle sea $s \in \overline{S}_{n,n}$.

Algorithm.

While there is a ball in the nonpositive quarter
  Choose the ball of coordinates (x, y) with
  $x = \min_i\{(i, y) \in s;\ \forall y\}$, $y = \max_i\{(x, i)\}$
  If there is a square on the zero-line
    Find the square of coordinates $(x', 0)$ with
    $x' = \min_i\{(i, 0) \in s;\ x' > x\}$
    $i \leftarrow x' - x$
    Add a part that is equal i to
      α if $y = 0$
      β otherwise
    Delete the ball
    Change the square into a ball
  Else
    Find the square of coordinates (x', y') with
    $x' = \max_i\{(i, y) \in s;\ \forall y\}$,
    $y' = \max_i\{(x', i)\}$
    $i \leftarrow x' - x$
    Add a part i to
      γ if $y = 0$
      δ otherwise
  Delete the ball and the square

A careful look at the algorithm shows that α is a nonincreasing sequence, β an decreasing sequence, γ a decreasing sequence and δ nonincreasing sequences. For a detailed proof, see [4].

Example. Starting with the sea $s \in \overline{S}_{6,6}$ on Fig. 1 we get that $n = 6$ and

$$\alpha = (6, 3, 2, 1), \qquad \beta = (5, 3), \qquad \gamma = (5, 4, 1), \quad \text{and} \quad \delta = (2).$$

The sea s is such that $|s| = 38$, $d_0(s) = 2$, $g_0(s) = 3$, $t_g(s) = 4$. We can check that

$$|\alpha| + |\beta| + |\gamma| + |\delta| = 38 = |s|, \qquad l(\alpha) + l(\beta) + l(\gamma) + l(\delta) = 10 = n + t_g(s),$$
$$l(\beta) + l(\delta) = 3 = n - g_0(s), \qquad l(\gamma) + l(\delta) = 4 = n - d_0(s).$$

6. Ramanujan's ${}_1\psi_1$ summation

Let us rewrite Ramanujan's ${}_1\psi_1$ summation:

$$\frac{(-aq;q)_\infty(-bq;q)_\infty}{(q;q)_\infty(abq;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-a^{-1};q)_n(zqa)^n}{(-bq;q)_n} = \frac{(-zq;q)_\infty(-z^{-1};q)_\infty}{(bz^{-1};q)_\infty(azq;q)_\infty}.$$

Let

$$A_m = [z^m]\frac{(-zq;q)_\infty(-z^{-1};q)_\infty}{(bz^{-1};q)_\infty(azq;q)_\infty}.$$

As explained in [4], the fact that

$$A_0 = \frac{(-aq;q)_\infty(-bq;q)_\infty}{(q;q)_\infty(abq;q)_\infty}$$

follows from the previous bijection. Therefore, we need to show for any m

$$A_0\frac{(-a^{-1};q)_m(aq)^m}{(-bq;q)_m} = A_m.$$

Using particle seas we will first prove that for $m > 0$

$$A_0\prod_{i=1}^{m}\bigl(aq + q^i\bigr) = \prod_{i=1}^{m}\bigl(1 + bq^i\bigr)A_m.$$

We define for $m > 0$ and $n \geqslant 0$ the bijection

$$\Psi_m : \widetilde{S}_0 \times D_m \to \widetilde{S}_m \times D_m.$$

This bijection is such that if $(\sigma, \beta) = \Psi_m(s, \alpha)$ then:

$$|s| + |\alpha| + m - l(\alpha) = |\sigma| + |\beta|, \qquad g(s) - g_0(s) = g(\sigma) - g_0(\sigma) + l(\beta),$$
$$m - l(\alpha) + d(s) - d_0(s) = d(\sigma) - d_0(\sigma).$$

This bijection Ψ_m and its reverse are now explained starting with a particle sea $s \in \widetilde{S}_0$ and $\alpha \in D_m$.

Algorithm.

$\Psi_m : (s, \alpha) \mapsto (\sigma, \beta)$
$\sigma \leftarrow s$
For i from 1 to m do
  If $i \in \alpha$ then
    If there exists a ball $(0, x)$, $x > 0$
      Delete it
      Add a part of size i to β
    Else translate the integer line by 1
  Else add a square of abscissa 1

$\Psi_m^{-1} : (\sigma, \beta) \mapsto (s, \alpha)$
$s \leftarrow \sigma$
For i from 1 to m do
  If $i \in \beta$ then
    Add a ball of abscissa 0
    Add a part of size i to α
  Else
    If there exists a square $(1, x)$, $x > 0$
      Delete it
    Else
      Add a part of size i to α
      Translate the integer line by -1

From that definition it is easy to see that Ψ_m is a bijection and that all the conditions are satisfied.

Example. We will do this example using the notation of overlined partitions. Starting with $s = ((\bar{6}, 5, 5, \bar{4}, 4, 1, \bar{0}), (\bar{3}, 3, 3, \bar{2}, \bar{1}, 1, 1))$ and $\alpha = (2, 1)$, we have $|s| = 39$, $g(s) = d(s) = 7$, $g_0(s) = d_0(s) = 3$, $|\alpha| = 3$, and $l(\alpha) = 2$. Applying Ψ_3 gives:

$$i = 1,\ i \in \alpha \quad \Rightarrow \quad \sigma = \big((\bar{5}, 4, 4, \bar{3}, 3, 0), (\bar{4}, 4, 4, \bar{3}, \bar{2}, 2, 2)\big), \qquad \beta = (\,),$$
$$i = 2,\ i \in \alpha \quad \Rightarrow \quad \sigma = \big((\bar{5}, 4, 4, \bar{3}, 3), (\bar{4}, 4, 4, \bar{3}, \bar{2}, 2, 2)\big), \qquad \beta = (2),$$
$$i = 3,\ i \notin \alpha \quad \Rightarrow \quad \sigma = \big((\bar{5}, 4, 4, \bar{3}, 3), (\bar{4}, 4, 4, \bar{3}, \bar{2}, 2, 2, 1)\big), \qquad \beta = (2).$$

See Fig. 5.

We have $|\sigma| = 41$, $g(\sigma) = 5$, $d(\sigma) = 8$, $g_0(\sigma) = 2$, $d_0(\sigma) = 3$, $|\beta| = 2$, and $l(\beta) = 1$. Note that the conditions are satisfied:

$$|s| + |\alpha| + m - l(\alpha) = 43 = |\sigma| + |\beta|, \qquad g(s) - g_0(s) = 4 = g(\sigma) - g_0(\sigma) + l(\beta),$$
$$d(s) - d_0(s) + m - l(\alpha) = 6 = d(\sigma) - d_0(\sigma).$$

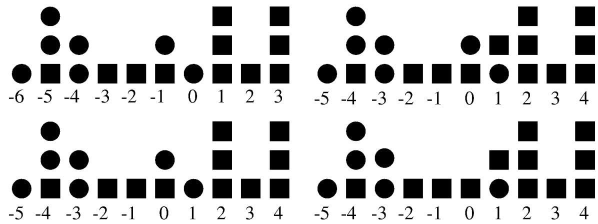

Fig. 5. Example of Ψ_3.

To finish the proof we need to show that for $m > 0$

$$A_0 \prod_{i=0}^{m-1} \left(b + q^i\right) = \prod_{i=1}^{m} \left(1 + aq^i\right) A_{-m}.$$

We define for $m > 0$ the bijection

$$\Psi_{-m} : \widetilde{S}_0 \times D_{m-1,\geqslant} \to \widetilde{S}_m \times D_m.$$

This bijection is such that if $(\sigma, \beta) = \Psi_{-m}(s, \alpha)$ then:

$$|s| + |\alpha| = |\sigma| + |\beta|, \qquad g(s) - g_0(s) + m - l(\alpha) = g(\sigma) - g_0(\sigma),$$
$$d(s) - d_0(s) = d(\sigma) - d_0(\sigma) + l(\beta).$$

This bijection Ψ_{-m} and its reverse are now explained.

Algorithm.

$\Psi_{-m} : (s, \alpha) \mapsto (\sigma, \beta)$
$\sigma \leftarrow s$
For i from 0 to $m - 1$ do
  If $i \in \alpha$ then
    If there exists a square $(1, x)$, $x > 0$
      Delete it
      Add a part of size $i + 1$ to β
    Else translate the integer line by -1
  Else add a ball of abscissa 0

$\Psi_{-m}^{-1} : (\sigma, \beta) \mapsto (s, \alpha)$
$s \leftarrow \sigma$
For i from 1 to m do
  If $i \in \beta$ then
    Add a square of abscissa 1
    Add a part of size $i - 1$ to α
  Else
    If there exists a ball $(0, x)$, $x > 0$
      Delete it
    Else
      Add a part of size $i - 1$ to α
      Translate the integer line by 1

Example. We will do this example using the notation of overlined partitions. Starting with $s = (\bar{6}, 5, 5, \bar{4}, 4, 1, \bar{0}), (\bar{3}, 3, 3, \bar{2}, \bar{1}, 1, 1))$ and $\alpha = (2, 1)$. We have $|s| = 39$, $g(s) = d(s) = 7$, $g_0(s) = d_0(s) = 3$, $|\alpha| = 3$, and $l(\alpha) = 2$. Applying Ψ_{-3} gives:

$$\begin{aligned}
i=0,\ i\notin\alpha &\quad\Rightarrow\quad \sigma=\big((\bar 6,5,5,\bar 4,4,1,\bar 0,0),(\bar 3,3,3,\bar 2,\bar 1,1,1)\big), &\beta=(\,),\\
i=1,\ i\in\alpha &\quad\Rightarrow\quad \sigma=\big((\bar 6,5,5,\bar 4,4,1,\bar 0,0),(\bar 3,3,3,\bar 2,\bar 1,1)\big), &\beta=(2),\\
i=2,\ i\in\alpha &\quad\Rightarrow\quad \sigma=\big((\bar 6,5,5,\bar 4,4,1,\bar 0,0),(\bar 3,3,3,\bar 2,\bar 1)\big), &\beta=(3,2).
\end{aligned}$$

We have $|\sigma|=37$, $g(\sigma)=8$, $d(\sigma)=5$, $g_0(\sigma)=d_0(\sigma)=3$, $|\beta|=5$, and $l(\beta)=2$. Note that the conditions are satisfied:

$$|s|+|\alpha|=42=|\sigma|+|\beta|,\qquad g(s)-g_0(s)+m-l(\alpha)=5=g(\sigma)-g_0(\sigma),$$
$$d(s)-d_0(s)=4=d(\sigma)-d_0(\sigma)+l(\beta).$$

7. The other half of ${}_6\psi_5$

Now we give a bijective proof of (1.5):

$$\sum_{n=0}^{\infty}\frac{(1+dq^{2n})(-dq;q)_{n-1}(-1/b;q)_n(-1/c;q)_n(dbc)^n q^{n(n+1)/2}}{(q;q)_n(bdq;q)_n(cdq;q)_n}
=\frac{(-dq;q)_\infty(-dcbq;q)_\infty}{(dcq;q)_\infty(dbq;q)_\infty}$$

using the colored-square particle seas. This identity says using Lemma 6 that there is a bijection $\Gamma_n : C_{n,n}\to D\times D\times P\times P$ such that if $s\mapsto(\alpha,\beta,\gamma,\delta)$ then

$$l(\alpha)+l(\beta)+l(\gamma)+l(\delta)=n+t_g(s),\qquad l(\beta)=w(s),\qquad l(\gamma)=dg(s),$$
$$l(\delta)=gg(s),\qquad l(\alpha)=b(s).$$

This bijection is now explained starting with a particle sea $s\in\mathcal{C}_{n,n}$.

Algorithm.

While there is a square on the zero-line
  Choose the ball with coordinates $(x,0)$
  $x=\min_i\{(i,0)\in s\}$
  Find the square with coordinates $(x',0)$
  $x'=\min_i\{(i,0)\in s,\ x'>x\}$
  $i\leftarrow x'-x$
  Add a part i to
    β if the square is white
    γ if the square is dark gray
    δ if the square is gray
  Delete the ball
  Change the square into a ball

While there is a ball in the nonpositive quarter
  Choose the ball with minimum abscissa x
  Find the square with coordinates (x',y')
  $x'=\min_i\{(i,y)\in s,\ x'>x;\ \forall y\}$
  $y'=\max_i\{(x',i)\in s\}$
  $i\leftarrow x'-x$
  Add a part i to α
  Delete the ball and the square

A careful look at the algorithm shows that α and β are decreasing sequences, γ and δ nonincreasing sequences. Therefore $\alpha, \beta \in D$ and $\gamma, \delta \in P$. Then let us check that all the conditions are preserved. First we must have $l(\alpha) + l(\beta) + l(\gamma) + l(\delta) = n + t_g(s)$. Each ball in the nonpositive quarter creates a part (there are n balls) and each square in the nonpositive quarter is changed into a ball and therefore creates a part (there are $t_g(s)$ squares). Is is straightforward to see that $l(\beta) = w(s)$, $l(\gamma) = dg(s)$, $l(\delta) = gg(s)$, $l(\alpha) = b(s)$, as each white (respectively dark gray, gray, black) square is associated with a ball and creates a part in β (respectively γ, δ, α).

Example. Starting with the sea on Fig. 3, we get $\alpha = (8, 2)$, $\beta = (6, 1)$, $\gamma = (10, 2, 1)$, and $\delta = (10, 2, 1, 1)$.

Now we give the reverse bijection. Start with a particle sea which positive integer half line is filled with balls. We write α as an increasing sequence, β as a decreasing sequence, γ and δ nonincreasing sequences. We define an order $\beta > \gamma > \delta$ which means that if $\beta_1 = \gamma_1 \geqslant \delta_1$ (respectively $\beta_1 = \delta_1 > \gamma_1$ (respectively $\gamma_1 = \delta_1 > \beta_1$)) the largest part is β_1 (respectively β_1 (respectively γ_1)). Otherwise the largest part is $\max\{\beta_1, \gamma_1, \delta_1\}$.

Algorithm.

For i from 1 to $l(\alpha)$
  Extract α_1 (smallest part)
  Add a ball at $(-i+1, 0)$
  Add a black square at abscissa $(\alpha_1 - i + 1)$ and positive ordinate

For i from $1 + l(\alpha)$ to $l(\alpha) + l(\beta) + l(\gamma) + l(\delta)$
  Extract the largest part j
  Add a ball at $(-i+1, 0)$
  Add a square at $(j - i + 1, 0)$
  This square is
    white if the largest part was β_1
    dark gray if the largest part was γ_1
    gray if the largest part was δ_1

Example. Starting with $\alpha = (2, 8)$, $\beta = (6, 5)$, $\gamma = (10, 5)$, and $\delta = (10)$ we give the example of the reverse bijection on Fig. 6.

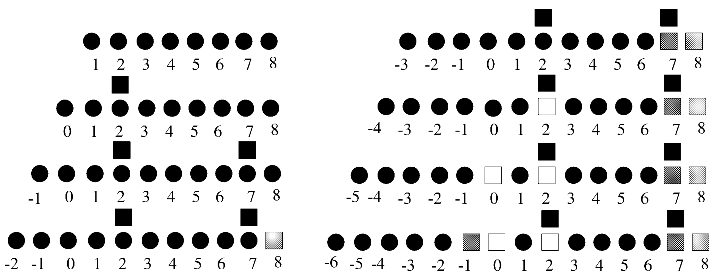

Fig. 6. Example of the reverse bijection.

Acknowledgments

We thank Xavier Viennot for his class and enthusiasm and Jeremy Lovejoy for his comments and advice during this work. We are indebted to the referees for a careful reading of the original manuscript and many helpful suggestions for its improvement.

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